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Domains, ranges and strategy-proofness: the case of single-dipped preferences

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Abstract We characterize the set of all individual and group strategy-proof rules on the domain of all single-dipped preferences on a line. For rules defined on this domain, and on several of its subdomains, we explore the implications of these strategy-proofness requirements on the maximum size of the rules' range. We show that when all single-dipped preferences are admissible, the range must contain two alternatives at most. But this bound changes as we consider different subclasses of single-dipped preferences: we provide examples of subdomains admitting strategy-proof rules with larger ranges. We establish exact bounds on the maximal size of strategy-proof functions on each of these domains, and prove that the relationship between the sizes of the subdomains and those of the ranges of strategy-proof functions on them need not be monotonic. Our results exhibit a sharp contrast between the structure of strategy-proof rules defined on subdomains of single-dipped preferences and those defined on subsets of single-peaked ones.

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1 Introduction

The existence of non-trivial strategy-proof rules depends both on the domain of preferences on which the rule is expected to operate, and on the range of alternatives that it is allowed to choose from. With only two alternatives on the range, different variants of strategy-proof rules can be defined, even for the universal domain (Barberà et al. 2011; Manjunath 2011). But if the range of a rule must contain three alternatives or more, then non-trivial (non-dictatorial) rules can only be strategy-proof if their domain is restricted appropriately (Gibbard 1973; Satterthwaite 1975). Thus, in principle, a mechanism designer in charge of proposing a social choice function may want to use two types of controls: one on the domain for which her design is intended, and one on the range that it should cover. In this paper, we show that these two strategic tools for design may not always be used independently. To do that, we concentrate on the domain of single-dipped preferences, and on its subdomains. We study the types of strategy-proof rules that one can construct on these sets of preferences, and show that once the choice of a single-dipped domain is made, the maximal size and form of their range is determined, in a rather tight manner. In particular, ranges will have to be typically "small". These results are in sharp contrast with the freedom that the designer is allowed for, in the choice of a range, when the domain of the function is a subset of single-peaked preferences. We shall elaborate on this general point later.

Let us now start to discuss single-dipped preferences. These preferences arise naturally in several contexts. One of them is attached to the existence of some public bad. Consider, for example, the decision on where to locate a facility whose neighborhood is undesirable, like a prison, a dumping site or an incineration plant. It is natural to assume that the worse allocation for each agent is the one that places the facility right by their home, and that locations become better as they place it further away. When the location of individual homes and of the facility can be identified with points on a line, fixing a home location gives rise naturally to single-dipped preferences on the facility location.

The same interpretation of points in a line as locations provides an argument for single-peaked preferences, a much more studied and somewhat dual case that arises, in the case when proximity to the public facility is desirable, rather than bad.

In fact, these two types of preferences also arise naturally from assumptions on the fundamentals of very simple models, other than the location example we just started with. For example, when agents with linear preferences must choose from the downward sloping frontier of a set of feasible alternatives in a two-good model (see Figs. 1a and 2a), one can identify this frontier with a segment of the line, over which individual preferences will be single-dipped, or single-peaked, depending on whether the frontier is convex or concave (see Figs. 1b and 2b).

The main purpose of this paper is methodological. We shall study different settings where the preferences of individuals belong to subsets of those that are single-dipped, and show that the ranges of strategy-proof rules on these domains must be severely restricted in size and shape. As a result, we can argue that controls over the domain and controls over the range of the functions they may propose are not tools that a designer can always use independently.

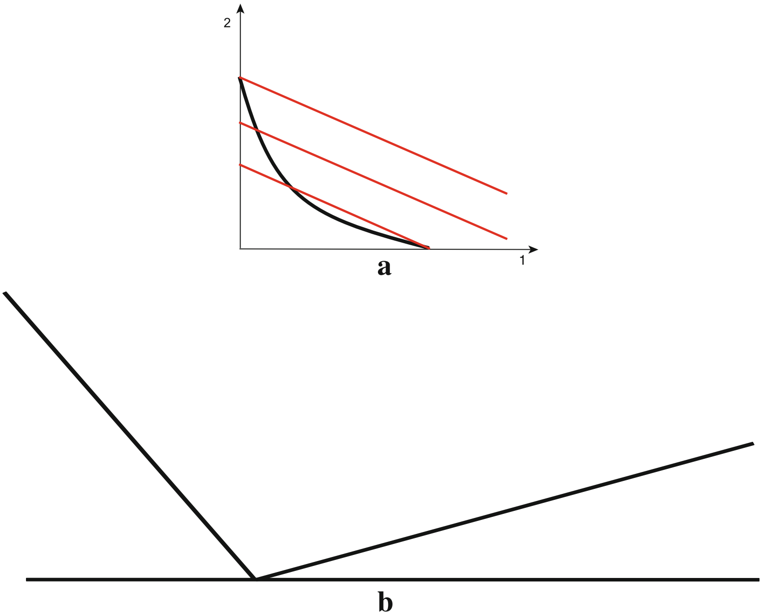


Fig. 1 **a** Linear preferences over a convex frontier. **b** Induced single-dipped preferences by linear preferences over a convex frontier

In addition to and beyond making this methodological point, we also hope that our insights on the form of strategy-proof rules defined on different subsets of single-dipped preferences can provide guidance for the choice of specific ones to be used. Indeed, we do provide a menu of rules that designers can choose from, when coping with the sort of locational or distributional issues that we just described, in contexts where the preferences of individuals conform to the single-dipped pattern. Moreover, we provide examples of specific subsets of single-dipped preferences that we find particularly attractive.

It is natural to compare the rules that are strategy-proof over single-dipped domains with those that satisfy the same condition for single-peaked domains. The set of all strategy-proof rules whose domain includes all single-peaked preferences was characterized by Moulin (1980). These rules, called generalized median voter schemes, constitute a rich class and contain many alternative procedures. The present paper contains a characterization of those rules that are strategy-proof when the domain includes all single-dipped preferences.[1]

An important feature that is common in both cases is that all rules that are strategy-proof on each of these domains, is also group strategy-proof. This is because both satisfy a condition called sequential inclusion (see Barberà et al. 2010) that

[1] There are other characterizations: see Peremans and Storcken (1999), Manjunath (2010).

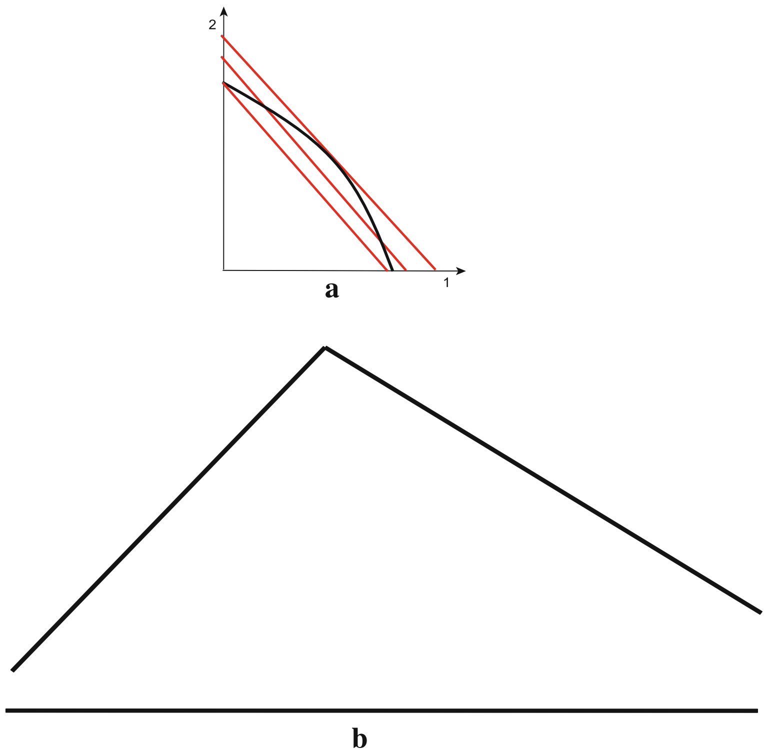


Fig. 2 **a** Linear preferences over a concave frontier. **b** Induced single-peaked preferences by linear preferences over a concave frontier

guarantees the equivalence of these two otherwise different incentive-compatibility requirements.[2]

However, this coincidence regarding the equivalence of individual and group strategy-proofness does not carry over other characteristics of our rules on these two domains. In particular, they dramatically differ regarding the characteristics of their ranges. In the case of single-peaked preferences and their subdomains, the range of strategy-proof functions can often consist of the whole set of alternatives.[3] By contrast, we will show that in domains where all single-dipped preferences are feasible, the range of strategy-proof rules contain at most two alternatives. This striking limitation is only one instance of a more general fact: that the size of the maximal ranges for strategy-proof rules on families of single-dipped preferences is endogenously predetermined by the nature of the domains that it must be defined on. This leads us to consider different subdomains of single-dipped preferences, to establish the maximal

[2] Other analogies between single-dipped and single-peaked domains were emphasized in Saari and Valognes (1999). As we shall see, our emphasis is on their differences, that are also substantial on some accounts.

[3] This is true for rules defined on the full domain of single-peaked preferences, and for many other subdomains. To get a full range it is sufficient (though not necessary) that any alternative is top for each agent at some admissible preference.

sizes that the range of strategy-proof rules over them, and to exhibit examples of rules where these sizes, that can certainly be larger than two, are effectively attained.

Of course, all strategy proof rules on the domain of all single-dipped preferences are also strategy-proof on its subdomains. But as these become more restrictive, new strategy-proof rules may arise. Some of those that we describe are of special interest, and all of them will still be group strategy-proof, as a result of our already mentioned equivalence result.

The work of Peremans and Storcken (1999) is an important predecessor of ours. Indeed, they already pointed at the equivalence between individual and group strategy-proofness in subdomains of single-dipped preferences, a phenomenon that we can rationalize and extend because we now can check that the condition of sequential inclusion is satisfied on any such subset of profiles. Peremans and Storcken started a systematic study of restrictions imposed by strategy-proofness on the ranges of rules defined for special subdomains of single-dipped preferences. We improve on the bound that they propose and we analyze several new cases for which we can also provide tight results. However, there is no denial that theirs is a pioneer study of the subject.

A recent paper by Manjunath (2010) provides a result that is very similar to the one we obtain for the domain of all single-dipped preferences, in that it also shows that the range of rules must be of size two, and also provides a characterization of all strategy-proof rules in that case. The main differences are that, unlike Manjunath, we do not impose the requirement that rules are unanimous and we do not concentrate on a bounded interval in the real line, and this allows us to be slightly more general on those points. Another difference is that we use a result of our own in the characterization of the rules (see Barberà et al. 2011), while he appeals to a previous result by Larsson and Svensson (2006). In that specific aspect, our paper and Manjunath's seem to be nicely complementary. After that our contribution, as already explained, takes the direction of exploring new subdomains and to provide additional results on the maximal sizes of ranges allowed by the strategy-proofness requirement.

Other literature where single-dipped preferences and strategy-proofness are studied refers to the provision of private goods, either in the problem of allocating an infinitely divisible good among agents (see Klaus et al. 1997 and Klaus 2001) or in assignment problems of an indivisible object (see Klaus 2001). Ehlers (2002) extends the deterministic model and allow allocation rules to be probabilistic.

The paper is organized as follows. Section 2 contains the model and definitions while Sect. 3 encompasses the results concerning the set of all single-dipped preferences. Finally, in Sect. 4 we gather some examples of rules for subdomains of single-dipped preferences and also the results concerning the size of the range of strategy-proof rules on such subdomains.

2 The setup and definitions

Let A be a finite set of *alternatives*[4] and $N = \{1, \ldots, n\}$ be a finite set of *agents*. Let $\mathcal{R}$ denote *the set of admissible preferences for any agent* $i \in N$, such that individual

[4] All results in Sect. 3 hold if A is any closed interval in the real line or the real line itself.

preferences are preorders (complete, reflexive, and transitive binary relations on A). We denote by $R_i \in \mathcal{R}$ an admissible preference relation for agent i and let as usual, P_i and I_i be the strict and the indifference part of R_i, respectively. A *preference profile,* denoted by $R_N = (R_1, \dots, R_n)$, is an element of $\mathcal{R}^n = \mathcal{R} \times \cdots \times \mathcal{R}$. Let $C, S \subset N$ be two coalitions such that $C \subset S$ and c and s denote their cardinality. We will write the subprofile $R_S = (R_C, R_{S \setminus C}) \in \mathcal{R}^s$ when we want to stress the role of coalition C in S. Then the subprofiles $R_C \in \mathcal{R}^c$ and $R_{S\setminus C} \in \mathcal{R}^{s-c}$ denote the preferences of agents in C and in $S \setminus C$, respectively. When $S = N$, we simplify notation by using $(R_C, R_{N\setminus C})$ as (R_C, R_{-C}).

We now define formally the notion of single-dipped preference relations relative to a given order of alternatives.

Definition 1 A preference relation of individual $i \in N$, R_i is single-dipped on A relative to a linear order $>$ of the set of alternatives if

(1) R_i has a unique minimal element $d_i(A)$, called the dip of i, and
(2) For all $y, z \in A$

$$[d_i(A) \geqslant y > z \text{ or } z > y \geqslant d_i(A)] \rightarrow \quad z P_i y.$$

Let $\mathcal{D}_>$ denote the set of all single-dipped preferences relations relative to $>$ while $\mathcal{R}_> \subseteq \mathcal{D}_>$ denote a subset of single-dipped preference relations relative to $>$. A preference domain of single-dipped preferences will be denoted by $\mathcal{R}^n_> = \times_{i \in N} \mathcal{R}_i$ where for each $i \in N$, $\mathcal{R}_i = \mathcal{R}_>$.[5]

Note that single-dipped preferences satisfy one of the forms of value restriction, as defined in Sen and Pattanaik (1969).

A *social choice function* (or a rule) is a function $f : \mathcal{R}^n_> \rightarrow A$. Let A_f denote the range of the social choice function f.

We will focus on rules that are nonmanipulable, neither by a single agent nor by a coalition of agents. We first define what we mean by a manipulation and then we introduce the well known concepts of *strategy-proofness* and *group strategy-proofness*.

Definition 2 A social choice function f is group manipulable on $\mathcal{R}^n_>$ at $R_N \in \mathcal{R}^n_>$ if there exists a coalition $C \subset N$ and $R'_C \in \mathcal{R}^c_>$ ($R'_i \neq R_i$ for any $i \in C$) such that $f(R'_C, R_{-C}) P_i f(R_N)$ for all $i \in C$. We say that f is individually manipulable if there exists a possible manipulation where coalition C is a singleton.[6]

Definition 3 A social choice function f is group strategy-proof on $\mathcal{R}^n_>$ if f is not group manipulable for any $R_N \in \mathcal{R}^n_>$. Similarly, f is strategy-proof if it is not individually manipulable.

5 We define single-dipped domains as cartesian products of sets of individually single-dipped preferences, all of them relative to the same order. The cartesian product structure is essential for the analysis of strategy-proofness. The additional implicit assumption that the sets of admissible preferences for all agents are identical is not essential, but allows us to lighten notation and provides enough richness for our purposes.

6 Our definition requires that all agents in a coalition that manipulates should obtain a strictly positive benefit from doing so. We consider this requirement compelling, since it leaves no doubt regarding the incentives for each member of the coalition to participate in a collective deviation from truthful revelation. For the analysis of a stronger version of group strategy-proofness in the present setting, see Manjunath (2010).

Notice that the domains of our social choice functions will always have the form of a cartesian product. This is necessary to give meaning to our definitions of individual and group strategy-proofness.

Barberà et al. (2010) showed that any subset of single-dipped preferences profiles satisfies a domain condition called sequential inclusion. They also showed that for domains satisfying such condition, strategy-proofness and group strategy-proofness turn out to be equivalent (see next Remark 1). From now on, we will use strategy-proofness and group strategy-proofness indistinctly.

Remark 1 (See Theorem 1 in Barberà et al. 2010) Any strategy-proof rule f defined on $\mathcal{R}^n_> \subseteq \mathcal{D}^n_>$ is group strategy-proof.

3 Strategy-proofness on $\mathcal{D}^n_>$

In this section we provide a characterization of all strategy-proof rules on the set of all single-dipped preferences. After reminding the reader of some facts that are relevant for our purposes, we establish that the range of these functions must contain at most two alternatives. That is, all non-constant strategy-proof rules on the domain should establish which pair of preselected alternatives prevails. We then combine this fact with a characterization result on strategy-proof rules with range two, in order to get the characterization.

We start by stating a well-known result that applies for any domain of preferences, not only for single-dipped ones. We include its straightforward proof, for the sake of completeness.

Definition 4 Let $\times_{i \in N}\mathcal{R}_i \subseteq \mathcal{R}^n$ such that $\mathcal{R}_i$ may differ from $\mathcal{R}_j$ for any $i, j \in N$. A social choice function f is Pareto efficient on A_f if for any $R_N \in \times_{i \in N}\mathcal{R}_i \subseteq \mathcal{R}^n$ there is no alternative $x \in A_f$ such that $x P_i f(R_N)$ for all $i \in N$.

Lemma 1 *Any group strategy-proof social choice function f on $\times_{i \in N}\mathcal{R}_i$ is Pareto efficient on the range.*[7]

Proof By contradiction suppose there exist $R_N \in \times_{i \in N}\mathcal{R}_i$ and $x \in A_f$ such that $x P_i f(R_N)$ for all $i \in N$. Let R'_N such that $x = f(R'_N)$. Let $S \subset N$ be the set of agents i such that $R_i \neq R'_i$. Note that $S \neq \varnothing$. Then, S manipulates f at R_N via R'_S which contradicts group strategy-proofness. □

Peremans and Storcken (1999) show that as a consequence of their Lemmas 3 and 4, any given strategy-proof rule on any subdomain of single-dipped preferences has at most 2^n alternatives in the range. In particular this upper bound is determined by the cardinality of the set of admissible profiles when each agent has only two admissible preferences (leftist, i.e., the order of alternatives according to the "smaller than" relation and rightist, i.e., the order of alternatives according to the "greater than" relation). This upper bound can be attained for this very restricted domain provided that there are

[7] The definition of group strategy-proofness is valid for any cartesian product of individual domain of preferences.

enough feasible alternatives. We obtain other bounds for the size of the range, which depend on the size and nature of the preferences that constitute admissible domains. The first bound applies when all single-dipped preferences are admissible and it is expressed in Theorem 1: only two alternatives may be in the range! Other interesting bounds for the sizes of ranges under different subdomains of single-dipped preferences are obtained in Sect. 4. In the next Theorem 1 we refine such Peremans and Storcken (1999)'s result by obtaining a more accurate upper bound when all single-dipped preference profiles are admissible.

Theorem 1 *Any strategy-proof social choice function f on $\mathcal{D}^n_>$ is such that $\#A_f \leq 2$.*

Note, as we show in the following example, that the result in Theorem 1 can not be generalized to all subsets of single-dipped preferences. In the next section we stress this point.

Example 1 Let $N = \{1, 2\}$, $A = \{x, y, z\}$ where $z > y > x$ and the set of individual admissible preferences is $\mathcal{R}_> = \{R, R', \overline{R}\}$ where $xPyPz$, $xP'zP'y$, and $z\overline{P}y\overline{P}x$. Note that $\mathcal{R}^2_>$ is a subset of single-dipped profiles relative to the above defined order of alternatives. The social choice function f on $\mathcal{R}^2_>$ defined as in Equation (1) is strategy-proof and the size of its range ($\#A_f$) is $3 > 2$.

f	R_2	R'_2	$\overline{R}_2$
R_1	x	x	y
R'_1	x	x	z
$\overline{R}_1$	y	z	z

. (1)

Before proving Theorem 1, we introduce a list of relevant preferences over triples of alternatives, some useful notation, and other interesting results.

A list of relevant preferences. Notice that for any triple of alternatives in A, say t, formed by x, y, z where $z > y > x$, the restriction of any $R_N \in \mathcal{R}^n_> \subseteq \mathcal{D}^n_>$ to x, y, z, say $R_{N,t} = (R_{1,t}, \ldots, R_{n,t})$, takes the form of the following admissible relations (see list (2)):

$$\begin{array}{l} xP^1y, xP^1z, \quad \text{and} \quad yR^1z \\ xP^2zP^2y \\ xI^3z, \quad \text{and} \quad zP^3y \\ zP^4xP^4y \\ zP^5y, zP^5x, \quad \text{and} \quad yR^5x. \end{array} \tag{2}$$

We refer to such preferences by their superindex, calling each one of them type l preferences. Informally, we'll say that type 1 preferences are "leftist on t" and that type 5 preferences are "rightist on t". Note that type 1 preferences may represent in fact two possibilities, either yP^1z or else yI^1z. Similarly, for type 5 preferences we may have yP^5x or yI^5x.

Notation: Take a triple t formed by $x, y, z \in A$ and $S \subset N$. For any $l \in \{1, 2, 3, 4, 5\}$, denote as $R^{l,t}_S$ any subprofile of preferences of agents in S, where for any $j \in S$, $R^{l,t}_j$ is such that its restriction to $\{x, y, z\}$ coincides with R^l in list (2) above.

Let us point out that the following three results apply for any subdomain of single-dipped preferences profiles. The first lemma states a property that any subdomain of single-dipped preferences must satisfy. Lemma 2 guarantees, when considering strategy-proof rules, the existence of two individual preferences for any triple t of alternatives in the range, one that is "leftist on t" and one that is "rightist on t".

Lemma 3 ensures that any alternative in the interior of the range can be obtained as the outcome of a preference profile where individual preferences are either "leftist on t" or "rightist on t".

Lemma 2 *Let f be a strategy-proof social choice function on $\mathcal{R}^n_> \subseteq \mathcal{D}^n_>$ with $\#A_f \geq 3$. For any triple t formed by $x, y, z \in A_f$ such that $z > y > x$, there exist $R^{1,t}$ and $R^{5,t} \in \mathcal{R}_>$.*

Proof Fix a triple t: $x, y, z \in A_f$ such that $z > y > x$, and let $x = f(R'_N)$, $y = f(\widetilde{R}_N)$, and $z = f(\overline{R}_N)$. Suppose first that there does not exist any type 1 preference on t. That is, for any $R_N \in \mathcal{R}^n_>$ only preferences of type 2, 3, 4, and 5 may coexist. Then, N would manipulate f at $\widetilde{R}_N$ via $\overline{R}_N$ since $z = f(R_N)\widetilde{P}_i f(\widetilde{R}_N) = y$ for any $i \in N$ which is a contradiction to group strategy-proofness. Thus, there exist $R_N \in \mathcal{R}^n_>$ such that restricted to t there are type 1 preferences.

Suppose that there does not exist any type 5 preference on t. That is, for any $R_N \in \mathcal{R}^n_>$ only preferences of type 1, 2, 3, and 4 may coexist. Then, N would manipulate f at $\widetilde{R}_N$ via R'_N since $x = f(R'_N)\widetilde{P}_i f(\widetilde{R}_N) = y$ for any $i \in N$ which is a contradiction to group strategy-proofness. Thus, there exist $R''_N \in \mathcal{R}^n_>$ that are type 5 preferences when restricted to t. This ends the proof. □

Lemma 3 *Let f be a strategy-proof social choice function on $\mathcal{R}^n_> \subseteq \mathcal{D}^n_>$ with $\#A_f \geq 3$. For any triple t: $x, y, z \in A_f, z > y > x$, then $y = f(R^{1,t}_S, R^{5,t}_{N\setminus S})$ for some $S \subseteq N$.*

Proof Let t be the triple $x, y, z \in A_f, z > y > x$, and $y = f(\widetilde{R}_N)$. Define $S_l = \{i \in N : \widetilde{R}_{i,t} = R^{l,t}\}$ for $l \in \{1, 2, 3, 4, 5\}$. Consider the set of agents in $N\setminus(S_1 \cup S_5)$ and define $\overline{S} = \{i \in N\setminus(S_1 \cup S_5) : d(\widetilde{R}_i) \geq y\}$. By strategy-proofness, $f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}}) = y$. The argument is as follows: observe first that $f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}}) \in [y, z)$, otherwise $\overline{S}$ would manipulate f at $\widetilde{R}_N$ via $R^{1,t}_{\overline{S}}$ and get an outcome strictly better than y. Second, $f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}}) = y$. Otherwise, if $f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}}) \in (y, z)$, $\overline{S}$ would manipulate f at $(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}})$ via $\widetilde{R}_{\overline{S}}$ and get y, since any $j \in \overline{S}$, $y P^{1,t}_j f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}})$. Thus, $f(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}}) = y$.

Define now $\widehat{S} = \{i \in N\setminus(S_1 \cup S_5) : d(\widetilde{R}_i) < y\}$. By strategy-proofness, $f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})}) = y$. The argument is similar to the one above: first note that $f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})}) \in (x, y]$. Otherwise, $\widehat{S}$ would manipulate f at $(R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus\overline{S}})$ via $R^{5,t}_{\widehat{S}}$ and get an outcome better than y for any $j \in \widehat{S}$. Second, $f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})}) = y$. Otherwise, if $f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})}) \in (x, y)$, $\widehat{S}$ would manipulate f at $(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})})$ via $\widetilde{R}_{\widehat{S}}$ and get y, since any $j \in \overline{S}$, $y P^{5,t}_j f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})})$. Thus, $f(R^{5,t}_{\widehat{S}}, R^{1,t}_{\overline{S}}, \widetilde{R}_{N\setminus(\overline{S}\cup\widehat{S})}) = y$. □

The following result obtained by the two previous lemmata assures that for any triple t: $z > y > x$ where $x, z \in A_f$, if alternative y is in the range then there can not exist simultaneously a type 2 and a type 4 individual preference over the triple t.

Theorem 2 *Let f be a non-constant strategy-proof social choice function on $\mathcal{R}^n_> \subseteq \mathcal{D}^n_>$. Let $x, z \in A_f$, and a triple $t : z > y > x$ for which there exists $R^{2,t}, R^{4,t} \in \mathcal{R}_>$. Then, $y \notin A_f$.*

Proof If $\#A_f = 2$ the result trivially holds. Suppose that $\#A_f \geq 3$. By contradiction suppose that $y \in A_f$. Let $x, z \in A_f$, and the triple t: $z > y > x$. By Lemma 2 there exist $R^{1,t}$ and $R^{5,t}$ in $\mathcal{R}_>$. Moreover, by Lemma 3, $y = f(R^{1,t}_S, R^{5,t}_{N\backslash S})$ for some $S \subseteq N$.

By group strategy-proofness, $f(R^{1,t}_S, R^{4,t}_{N\backslash S}) \in (y, z] \cup \{x\}$ (otherwise, coalition $N\backslash S$ would manipulate f at $(R^{1,t}_S, R^{4,t}_{N\backslash S})$ via $R^{1,t}_{N\backslash S}$. By Lemma 1, $f(R^{1,t}_S, R^{1,t}_{N\backslash S}) = x$) and again by group strategy-proofness, $f(R^{1,t}_S, R^{4,t}_{N\backslash S}) = x$ (otherwise, coalition $N\backslash S$ would manipulate f at $(R^{1,t}_S, R^{5,t}_{N\backslash S})$ via $R^{4,t}_{N\backslash S}$).

By group strategy-proofness, $f(R^{2,t}_S, R^{5,t}_{N\backslash S}) \in [x, y) \cup \{z\}$ (otherwise, coalition S would manipulate f at $(R^{2,t}_S, R^{5,t}_{N\backslash S})$ via $R^{5,t}_S$ and get $f(R^{5,t}_S, R^{5,t}_{N\backslash S}) = z$), and again by group strategy-proofness, $f(R^{2,t}_S, R^{5,t}_{N\backslash S}) = z$ (otherwise, coalition S would manipulate f at $(R^{1,t}_S, R^{5,t}_{N\backslash S})$ via $R^{2,t}_S$).

By group strategy-proofness, $f(R^{2,t}_S, R^{4,t}_{N\backslash S}) = z$ (otherwise, coalition $N\backslash S$ would manipulate f at $(R^{2,t}_S, R^{4,t}_{N\backslash S})$ via $R^{5,t}_{N\backslash S}$ and get $f(R^{2,t}_S, R^{5,t}_{N\backslash S}) = z$), but then coalition S would manipulate f at $(R^{2,t}_S, R^{4,t}_{N\backslash S})$ via $R^{1,t}_S$ and get $f(R^{1,t}_S, R^{4,t}_{N\backslash S}) = x$. Thus, we obtain a contradiction. □

Note that Theorem 2 generalizes Theorem 1 above. And in fact Theorem 1 is a straightforward corollary of Theorem 2. Observe that if $\mathcal{R}^n_> = \mathcal{D}^n_>$, for any t in A: $z > y > x$ there exist $R^{2,t}$ and $R^{4,t}$ in $\mathcal{R}_>$. Thus, recursively applying Theorem 2 we end up showing that the $\#A_f \leq 2$.

Observe that by Theorem 1, the following result straightforwardly holds.

Corollary 1 *There is no strategy-proof and onto social choice function on the domain of all single-dipped preferences if $\#A \geq 3$.*

In Barberà et al. (2011), we obtained a characterization of all strategy-proof rules with a binary range (that is, $A_f = \{x, y\}$ for some pair x, y of alternatives in A) by means of two conditions that we define below. Let $X(R_N) = \{i \in N : xP_i y\}$ and $Y(R_N) = \{j \in N : yP_j x\}$ for each preference profile $R_N \in \mathcal{R}^n_>$.

Definition 5 A social choice function f with a binary range is *essentially xy-monotonic* if and only if for all $R_N, R'_N \in \mathcal{R}^n_>$ such that $R_h = R'_h$, for any $h \in N\backslash[X(R_N) \cup Y(R_N)] \cap N\backslash[X(R'_N) \cup Y(R'_N)]$, the following holds:

(1) If $X(R'_N) \supseteq X(R_N)$, $Y(R_N) \supseteq Y(R'_N)$ (with at least one strict inequality), and $f(R_N) = x$, then $f(R'_N) = x$; and

(2) If $Y(R'_N) \supseteq Y(R_N)$, $X(R_N) \supseteq X(R'_N)$ (with at least one strict inequality), and $f(R_N) = y$, then $f(R'_N) = y$.

Definition 6 A social choice function f with a binary range is *essentially xy-based* if and only if for all $R_N, R'_N \in \mathcal{R}^n_>$ such that $X(R_N) = X(R'_N)$, $Y(R_N) = Y(R'_N)$, and $R_h = R'_h$, $\forall h \in N \backslash [X(R_N) \cup Y(R_N)]$, then $f(R_N) = f(R'_N)$.

These two conditions characterize strategy-proof social choice functions with a binary range.

Theorem 3 (see Barberà et al. 2011) *Let f be a social choice function on $\mathcal{R}^n_> \subseteq \mathcal{R}^n$ with a binary range. Then, f is strategy-proof if and only if f is essentially xy-monotonic and essentially xy-based.*

We already know, from our previous work, that essentially xy-monotonicity and the essentially xy-based condition are independent whatever the domain of preferences is. For the sake of completeness, in the following example we present two rules violating only one of them when we concentrate on the set of single-dipped preferences profiles.

Example 2 Let $N = \{1, 2\}$, $A = \{x, y, z\}$ where $z > y > x$, and for any $i \in N$, the set of individual admissible preferences is $\mathcal{R}_> = \{R^1, R^2, R^4, R^5\}$ where xP^1yP^1z, xP^2zP^2y, zP^4xP^4y and zP^5yP^5x. Note that $\mathcal{R}^2_>$ is the subset of all strict single-dipped profiles relative to the above defined order of alternatives. Observe that f defined in Eq. 3 satisfies essentially xz-based but it violates essentially xy-monotonicity. Note also that $\widehat{f}$ defined in Eq. 4 satisfies essentially xy-monotonicity but it violates essentially xz-based. Both f and $\widehat{f}$ are manipulable.

f	R_2^1	R_2^2	R_2^4	R_2^5
R_1^1	z	$\boxed{z}$	x	x
R_1^2	z	z	$\boxed{x}$	x
R_1^4	x	x	x	x
R_1^5	x	x	x	x

(3)

$\widehat{f}$	R_2^1	R_2^2	R_2^4	R_2^5
R_1^1	$\boxed{z}$	$\boxed{x}$	z	z
R_1^2	x	x	z	z
R_1^4	z	z	z	z
R_1^5	z	z	z	z

(4)

Therefore, as a corollary of Theorems 3 and 1, we can state the following result.

Proposition 1 *Let f be a social choice function on $\mathcal{D}_>$ with a binary range. Then, f is strategy-proof if and only if f is essentially xy-monotonic and essentially xy-based.*

4 Strategy-proofness on restricted single-dipped domains

We now present three families of subdomains of single-dipped preferences. For two of them we describe classes of rules that are strategy-proof on these domains, and yet have ranges of size larger than two. For the third one, we show that even it is a subset of one of those two, we are back to the situation where the range of any strategy-proof rule on it can only contain at most two alternatives.

Our analysis is not exhaustive. We do not work on all possible subdomains of single-dipped preferences, nor do we claim that the rules we exhibit exhaust the set of all those satisfying strategy-proofness on the respective domains. Our aim is a more limited one, hopefully interesting for the reader. It is to show that there is no inherent association between the fact that preferences are single-dipped and the need for the range to be limited to size two, nor between this limitation of the range and the fact that only two alternatives can be the best for agents, under single-dipped preferences. These connections are only effective when all single-dipped preferences are in the domain of our rules.

We present our examples for the case where domains consist of preferences defined on a finite set of alternatives. With some changes and qualifications, our examples and results could also extend to the case where the set of alternatives is a continuum. However, as already remarked by Peremans and Storcken (1999), the ranges will still be finite in that case. Again, this is a substantial difference with the case of single-peakedness.

Before we formally present our subdomains, let us informally describe the kind of preferences that they try to capture. As we have already said, their use in the paper is instrumental, as examples. Yet, we like to stress that they represent meaningful and attractive types of preferences, that may arise in applications.

We present the underlying idea by referring to preferences on a line. Clearly, single-dipped preferences correspond to agents whose best alternatives are extreme, and that may well swing from one end of the spectrum to the other. A preference where one end of the line is best and the other end of the line is worse can be single-dipped. In fact, nothing precludes that the closest alternative to the peak may be the dip of that preference. What our restrictions avoid are these extreme swings in preferences, by introducing a systematic bias in favor of alternatives that lie "close" and "on the same side" than the agent's peak. For example, a preference with bias 3 would be one where, should one end of the line be the peak, the next best 3 alternatives in the order would be those that are immediately contiguous to the best one. So, the position of the dip would be bounded away from that of the peak. In addition, the preferences of agents would only "swing" to the other end of the line after a certain number of alternatives in the neighborhood of the peak have already been ranked above.

This same underlying idea gives rise to domains with different properties, depending on the size of the bias relative to that of the total number of alternatives. If the bias is in favor of a "small" number of alternatives (less than half the total number of possible choices), then there will be locations that can be the dips of voters whose peak is in either of the extremes. Whereas, if the bias is large enough, then the set of possible dips for people for peaks on one end of the line, and for people on the other end are disjoint. It turns out, as we shall see, that this has important consequences on the potential ranges of strategy-proof functions, whose size will be different in one case and in the other.

We shall now turn to formal definitions. But let us emphasize again that single-dipped domains with a bias reflect a rather natural phenomenon: that of agents who try to avoid intermediate positions, but still have a distinctive preference for one end of the line than for the other. These preferences may well arise in applications.

We will let the set of alternatives be the integer interval $A = [0, 1, 2, \ldots, k]$, $k \geq 3$. For $k \leq 2$, the subdomains we define below would give rise to non-interesting results. We denote by $[a, b]$ the set of integer numbers between a and b both included, while (a, b) excludes both a and b.

Definition 7 For any given set of alternatives of size $k+1$, and any integer h, $0 < h < k$, the set $\mathcal{D}_{kh}$ of single-dipped preferences with an h bias consists of all preferences R that:

(1) are single-dipped, and
(2) either hPs for all $s \in [h+1, k]$, or else

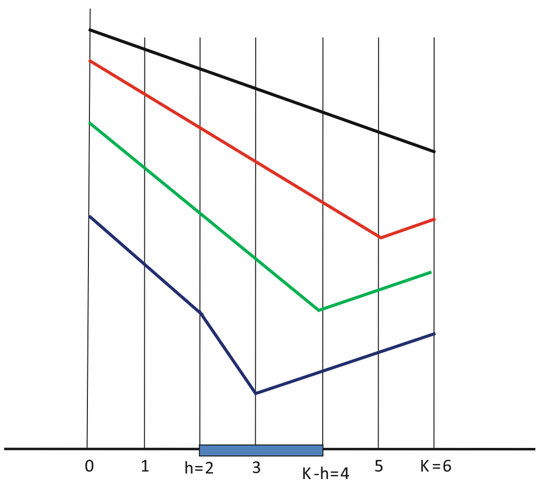
$(k-h)\ Ps$ for all $s \in [0, k-h-1]$.

Note first that condition (2) implies that alternatives 0 and k will not be indifferent. Notice also that if $h = 0$, condition (2) would have no bite and we would be back to the case of single-dipped preferences, where 0 and k are not indifferent. Moreover if $h = k-1$, then the only two admissible preferences are those where 0 is the best and k is the worst alternative, or its opposite.

In Figure 3 we present some preferences that satisfy our definition 7 for $k = 6$ and $h = 2$. We have chosen to represent some of those preferences that have 0 in first place, so that their dip cannot be in 1 or 2, and all other alternatives must be ranked below 2.

Notice also that, for any fixed k, $\mathcal{D}_{kh'} \subsetneq \mathcal{D}_{kh}$ where $0 < h < h' < k$. Furthermore, observe that if $h < \frac{k}{2}$, the set on which the dips of agents with tops in 0 and agents with tops on k can be located are overlapping, and their intersection is the segment $I^h = (h, k-h)$ whereas, if $h \geq \frac{k}{2}$, then the set I^h is always empty, and the preferences of all agents in the segment $C^h = [k-h, h]$ are strictly increasing or strictly

Fig. 3 Some single-dipped preferences with a bias $h = 2 < k/2, k = 6$



decreasing. These two subdomains will allow for different types of strategy-proof rules.

We now present, for each of these two domains, examples of group strategy-proof rules whose range size is larger than two. These examples are sufficient to prove that the bounds we establish later on are actually tight. We also feel that they provide strong hints toward what could be a more applied reading of our work here, as they point to the possible types of strategy-proof rules that can be defined in our subdomains. Indeed, they contain recipes for constructing rules that are strategy-proof and contain ranges "as large as possible" when the domain of preferences are biased and single-dipped. The reader can easily understand that, although we define a specific rule to serve our methodological purposes here, other similar rules can be easily constructed as variants of the ones we propose. Moreover, we believe that not much room is left, beyond these possible variants, to define any other rules that would be substantially different and satisfy the same properties.

Example 3 For $k \geq 3, 0 < h < \frac{k}{2}$, define a rule on $\mathcal{D}_{kh}$ as follows:
Fix a range consisting of four outcomes, r_1, r_2, r_3, r_4 such that $r_1 < r_2 \leq h$ and $r_3 > r_4 \geq k - h$.

For a given profile $R_N \in (\mathcal{D}_{kh})^n$, then

- Let $f(R_N) = r_1$ (respectively, r_2) if (1) $\#\{i \in N : r_2 P_i r_3\} \geq \#\{j \in N : r_3 P_j r_2\}$ and
  (2) $\#\{i \in N : r_3 P_i r_2 \text{ and } r_1 P_i r_2\} \geq \#\{j \in N : r_3 P_j r_2 \text{ and } r_2 P_j r_1\}$ (respectively, $<$).
- Let $f(R_N) = r_3$ (respectively, r_4) if (1) $\#\{i \in N : r_3 P_i r_2\} > \#\{j \in N : r_2 P_j r_3\}$, and
  (2) $\#\{i \in N : r_2 P_i r_3 \text{ and } r_3 P_i r_4\} \geq \#\{j \in N : r_2 P_j r_3 \text{ and } r_4 P_j r_3\}$ (respectively, $<$).

Informally, we could describe the rule as follows. Agents first vote by majority whether the outcome should be in $\{r_1, r_2\}$ or $\{r_3, r_4\}$, and then, those disagreeing with this majority vote again by majority to determine which of the two chosen alternatives should come out.

The rule is strategy-proof, and its range contains, by construction, four elements. The argument for (individual) strategy-proofness is as follows. Since preferences are single-dipped with a bias, all agents either prefer both r_1 and r_2 to both r_3 and r_4, or vice-versa. Hence, they will try to ensure that the outcome is any one of the two that they prefer, and no voter has a better strategy than supporting their best pair. In the second vote, agents who did not get their best alternatives pre-selected can still express their preferences between the other two, and either support the most extreme outcome or else get the less extreme one. Again, supporting their preferred alternative in this new binary vote is a dominant strategy. Group strategy-proofness is derived from Remark 1, that our domain satisfies sequential inclusion.

This rule provides an example of how, by restricting the domain of definition of our social choice functions, we may get group strategy-proof rules with a range larger than two. Remark that, contrary to what happened in richer domains, the rule we propose is strategy-proof but requires information beyond knowing what is the preferred alternative of each agent on the range. This is worth remarking because for many domains it is known that strategy-proof rules must only use information regarding the “tops on the range” of individual preferences. Our domains are such that this informational simplicity requirement can be skipped.

The choice of a range in this example is not capricious. The following theorem proves that, in the domain $\mathcal{D}_{kh}$ with $h < \frac{k}{2}$, this is the larger size of ranges admitting a strategy-proof rule.

Proposition 2 *Let $k \geq 3, 0 < h < \frac{k}{2}$. There is no strategy-proof social choice function $f : (\mathcal{D}_{kh})^n \to A$ with $\#A_f > 4$.*

Proof The proof consists of two steps.
Step 1: If $\#A_f \geq 3$, no alternative in the interval $(h, k-h)$ belongs to the range of f.
Proof of Step 1: Let $y \in (h, k-h)$ and let $x, z \in A_f$ such that $x < z$ without loss of generality. Observe that wherever the triple t: x, y, z, belongs to and for any possible order of them, there exist some preferences $R^{2,t}$ and $R^{4,t}$ in $\mathcal{D}_{kh}$.

If $z > y > x$, by Theorem 2, $y \notin A_f$.

If $z > x > y$ (respectively, $y > z > x$), if we assume that $y \in A_f$ then by Theorem 2 we obtain that $x \notin A_f$ (respectively, $z \notin A_f$) which is a contradiction. This shows Step 1.

Thus $A_f \subseteq [0, h] \cup [k-h, k]$.
Step 2: $\#\left(A_f \cap [0, h]\right) \leq 2$ and $\#\left(A_f \cap [k-h, k]\right) \leq 2$.
Proof of Step 2: Let us show that $\#\left(A_f \cap [k-h, k]\right) \leq 2$ (a similar argument would follow to show that $\#\left(A_f \cap [0, h]\right) \leq 2$). Suppose that there exists $S \subseteq [k-h, k] \cap A_f$ such that $\#S \geq 3$. Observe that for any triple in $[k-h, k]$, there exist some preferences $R^{2,t}$ and $R^{4,t}$ in $\mathcal{D}_{kh}$. Thus, fixed any pair $x^0, z^0 \in S$, $z^0 > x^0$, for any $y \in S$ such that $z^0 > y > x^0$, $y \notin A_f$ by Theorem 2. Repeatedly applying Theorem 2 for all different pairs $x^1, z^1 \in S$, $x^0 \geq x^1, z^1 \geq z^0$ we obtain that $\#S \leq 2$ which is a contradiction and shows Step 2.

Thus $\#A_f \leq 4$. □

Now, let us consider the following rules for the domain $\mathcal{D}_{kh}$ with $h > \frac{k}{2}$.

Example 4 For $k \geq 3$, $\frac{k}{2} < h < k-1$ define a rule on $\mathcal{D}_{kh}$ as follows:
Fix four outcomes, r_1, r_2, r_3, r_4 such that $0 \leq r_1, r_2 < k-h$ and $h < r_3, r_4 \leq k$.
For a given profile $R_N \in (\mathcal{D}_{kh})^n$,

- Let $f(R_N) = r_1$ (respectively, r_2) if (1) $0P_1k, 0P_2k$, and (2) $\#\{i \in N\backslash\{1,2\} : r_1 P_i r_2\} \geq$ (respectively, $<$)$\#\{j \in N\backslash\{1,2\} : r_2 P_j r_1\}$.
- Let $f(R_N) = r_4$ (respectively, r_3) if (1) kP_10, kP_20 and (2) $\#\{i \in N\backslash\{1,2\} : r_4 P_i r_3\} \geq$ (respectively, $<$) $\#\{j \in N\backslash\{1,2\} : r_3 P_j r_4\}$.
- Let $f(R_N) = \min\{h, k-h+\#\{i \in N\backslash\{1,2\} : kP_i0\}$ if $0P_1k$ and kP_20 or kP_10 and $0P_2k$.

Informally, we could describe the rule as follows. If agents 1 and 2 agree on their top then they select only two alternatives out of which the rest of agents will have to choose one. If agents 1 and 2 disagree on their top then the outcome will be an alternative within the interval $[k-h, h]$ and outcome will depend again on the preferences of the remaining agents (notice that their preferences on this interval will be either leftist or rightist). Let us briefly describe why this rule is strategy-proof. Since their preferences have an h bias, if 1 and 2 both prefer 0 to k, then they both prefer r_1 and r_2 to all other alternatives in the range. Hence if both agree that 0 is best, it is optimal for them to declare 0 and to obtain either r_1 or r_2. Similarly if both agree that k is best. And if agents 1 and 2 disagree, then it is best for both of them to avoid extreme outcomes r_1, r_2, r_3 and r_4 and to ensure through their sincere vote that the outcome lies in the range $[k-h, h]$. Given the votes of 1 and 2, the choices of the rest of the agents are either a binary election or else a vote on their best element on $[k-h, h]$ resulting in some outcome in this interval. In each of these three cases, being truthful is a dominant strategy for all them.

Let us remark that depending on k, h and n, the range of these functions can be as large as $\min\{2h-k+5, 2^n\}$.

In fact, this example provides an upper bound for the size of the range of strategy-proof rules on $\mathcal{D}_{kh}$ with $h > \frac{k}{2}$, as proven in the following proposition.

Proposition 3 *Let $k \geq 3$ and $\frac{k}{2} < h < k-1$. There does not exist any strategy-proof social choice function $f : (\mathcal{D}_{kh})^n \rightarrow A$ such that $\#A_f > \min\{2h-k+5, 2^n\}$.*

Proof Let $h \leq k-2$. We only need to show Step 1:
Step 1: $\#\left(A_f \cap (0, h)\right) \leq 2$ and $\#\left(A_f \cap (k-h, k)\right) \leq 2$.
If these two statements in Step 1 hold, then the maximum number of alternatives in the range is four plus the maximum number of alternatives in the range that belong to the interval $[k-h, h]$. That is, $\#A_f \leq 2+2+(h-[k-h-1]) = 2h-k+5$. This would end the proof of this Proposition.
Proof of Step 1: Let us show that $\#\left(A_f \cap (0, h)\right) \leq 2$ (a similar argument would follow to show that $\#\left(A_f \cap (k-h, k)\right) \leq 2$). Suppose that there exists $S \subseteq (0, h) \cap A_f$ such that $\#S \geq 3$. Observe that for any triple in $(0, h)$, there exist some preferences $R^{2,t}$ and $R^{4,t}$ in $\mathcal{D}_{kh}$. Thus, fixed any pair $x^0, z^0 \in S$, $z^0 > x^0$, for any $y \in S$ such that $z^0 > y > x^0$, $y \notin A_f$ by Theorem 2. Repeatedly applying Theorem 2 for all different pairs $x^1, z^1 \in S$, $x^0 \geq x^1$, $z^1 \geq z^0$ we obtain that $\#S \leq 2$ which is a contradiction. This shows Step 1. □

Finally, let us consider a third subdomain of preferences with an h bias, one that restricts the domain $\mathcal{D}_{kh}$ with $h < \frac{k}{2}$. Specifically, we will consider the set $\widehat{\mathcal{D}}_{kh}$:

Definition 8 Let $\widehat{\mathcal{D}}_{kh}$ be the domain formed by preferences that

(1) belong to $\mathcal{D}_{kh}$ with $h < \frac{k}{2}$ and
(2) $d(R_i) \in I^h = (h, k-h)$.

It turns out that, like in Theorem 1, it is now only possible to define strategy-proof rules on this subdomain of single-dipped preferences if the range is restricted to two alternatives at most, as shown by the following result.

Proposition 4 *Let $k \geq 3$, $0 < h < \frac{k}{2}$. There is no strategy-proof social choice function $f : \left(\widehat{\mathcal{D}}_{kh}\right)^n \rightarrow A$ with $\#A_f > 2$.*

Proof The proof consists of two steps.
Step 1: $\#\left(A_f \cap [0, h]\right) \leq 1$ and $\#\left(A_f \cap [k-h, k]\right) \leq 1$.
Proof of Step 1: Observe that for any $R \in \widehat{\mathcal{D}}_{kh}$, R is strictly decreasing in $[0, h]$ and strictly increasing in $[k-h, k]$. Suppose that there exist $x_1, x_2 \in A_f \cap [0, h]$ where $x_1 = f(R_N), x_2 = f(\widetilde{R}_N)$. Then, N would manipulate f: either at $\widetilde{R}_N$ via R_N if $x_1 < x_2$ or else at R_N via $\widetilde{R}_N$, otherwise. This is the desired contradiction. A similar argument holds for $A_f \cap [k-h, k]$. This shows Step 1.
Step 2: No alternative in the interval $(h, k-h)$ belongs to the range of f.
Proof of Step 2: Let $\#A_f \geq 3$ and by contradiction suppose that $A_f \cap (h, k-h) \neq \varnothing$. Suppose first that $A_f \subseteq (h, k-h)$. Then we get a contradiction to Theorem 2 since for any triple t: $x, y, z \in A_f$, there exist $R^{2,t}$ and $R^{4,t} \in \widehat{\mathcal{D}}_{kh}$ and thus $y \notin A_f$. Second, consider the case such that $A_f \cap ([0, h] \cup [k-h, k]) \neq \varnothing$ and $A_f \cap (h, k-h) \neq \varnothing$. We may have two subcases: (2.1) $x \in A_f \cap [0, h], z \in A_f \cap [k-h, k]$, and $y \in A_f \cap (h, k-h)$, and (2.2) $x \in A_f \cap [0, h], y, z \in A_f \cap (h, k-h)$ (or else $z \in A_f \cap [k-h, k], x, y \in A_f \cap (h, k-h)$). For both cases, consider the triple t: $x, y, z \in A_f$ and observe that there does not exist $R^{1,t}$ and $R^{5,t} \in \widehat{\mathcal{D}}_{kh}$ on the triple t which contradicts Lemma 2. Thus, the only possibility is that $A_f \subseteq [0, h] \cup [k-h, k]$ which shows Step 2.

Combining the results in both Steps, we obtain that $\#A_f \leq 2$. □

5 Conclusions

We have highlighted the fact that, in environments where preferences are single-dipped, bounds on the size of the ranges of social choice functions arise as necessary conditions for their strategy-proofness. We have shown how these bounds result from the interaction between the number of individuals and alternatives, and most importantly from the nature of the subdomains where the functions must be defined.

One important consequence of our research in this and other papers is a full characterization of the family of group strategy-proof rules on the full domain of single-dipped preferences.

The propositions on the bounds of the size of the ranges for functions defined on different subdomains also allow us to exhibit quite unexpected results. One is the very fact that such restrictions arise: this does not happen in other well studied domain restrictions admitting strategy-proof rules, like the domain of single-peaked or separable preferences (see Moulin 1980; Barberà et al. 1991). The other is that, in a very strong sense, the relationship between the size of single-dipped subdomains and that of the ranges of strategy-proof rules defined on them is not necessarily monotonic. We have exhibited one case where it is: this is the one where we compare the three domains $\mathcal{D}_{kh'} \subsetneqq \mathcal{D}_{kh} \subsetneqq \mathcal{D}_{>}$, for fixed k and $h < \frac{k}{2}, h' > \frac{k}{2}$. Those domains are nested, and the maximal range size compatible with strategy-proofness indeed increases for them as the domains shrink. However, we have also identified the three domains $\widehat{\mathcal{D}}_{kh} \subsetneqq \mathcal{D}_{kh} \subsetneqq \mathcal{D}_{>}$ for fixed k, and $h < \frac{k}{2}$. For these three nested domains, the admissible range size goes up, as we restrict the domain from $\mathcal{D}_{>}$ to $\mathcal{D}_{kh}$, and then goes down if we continue restricting the domain from $\mathcal{D}_{kh}$ to $\widehat{\mathcal{D}}_{kh}$. This definitely shows that the interactions we have unearthed are non-trivial.

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